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# Nonlinear superposition formulae for the differential-difference analogue of the KdV equation and two-dimensional Toda equation

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**Abstract.** In this paper, nonlinear superposition formulae of the differential-difference analogue of the KdV equation and two-dimensional Toda equation are proved rigorously. Some particular solutions of the differential-difference analogue of the KdV equation are given as an illustrative application of the obtained result.

## 1. Introduction

It is known that many integrable nonlinear equations share some common features, among which are the so called Backlund transformations (BTs). We can usually derive the nonlinear superposition formula from the commutability of BTs. Unfortunately, there is no rigorous proof for the commutability of BTs for a general nonlinear evolution equation [1, 2]. Therefore it is worthwhile to prove a nonlinear superposition formula directly. In 1978, Hirota and Satsuma [3] obtained simple nonlinear suerposition formulae in bilinear form of some celebrated equations such as KdV, MKdV, SG etc. Until now, some progress has been made in this field. However most work has only been done in $(1+1)$-dimensional nonlinear differential equations [4–14] and $(1+2)$-dimensional nonlinear differential equations [15–19]. Compared with the continuous case, the study of discrete integrable systems has received relatively less attention. In [3, 20, 21], different nonlinear superposition formulae for the Toda equation were considered. It is noted that recently Shabat *et al* have indicated a general connection between one-dimensional lattices with local symmetries and nonlinear integrable partial differential equations in $1+1$ dimensions (e.g. [22, 23], also see Levi's results [24]). A good example is provided by the Toda lattice representation of the nonlinear Schrödinger coupled equations [22]. Thus it seems to be more desirable to investigate discrete integrable systems directly. In this paper, we are going to prove the nonlinear superposition formulae of the differential-difference analogue of the KdV equation and two-dimensional Toda equation rigorously.

The content of this paper is organized as follows. In section 2, a nonlinear superposition formula of the differential-difference analogue of the KdV equation is shown. As an application of this result, some particular solutions of differential-difference analogue of the KdV equation are obtained. A BT and a nonlinear superposition formula for the two-dimensional Toda equation are described in section 3. In section 4, conclusions and a discussion are given. Finally we list some bilinear operator identities which are used in the paper in the appendix.

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## 2. Nonlinear superposition formula of the differential-difference analogue of the KdV equation

The differential-difference analogue of the KdV equation under consideration is [25]

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{w_n}{1+w_n}\right) = w_{n-1/2} - w_{n+1/2}. \tag{1}$$

By means of a variable transformation

$$w_n = \frac{\cosh(\frac{1}{2}D_n)f_n \cdot f_n}{f_n^2} - 1$$

(1) is reduced to the bilinear equation [26]

$$\sinh(\tfrac{1}{4}D_n)[D_t + 2\sinh(\tfrac{1}{2}D_n)]f_n \cdot f_n = 0 \tag{2}$$

where the bilinear operators are defined as follows [25, 27]

$$D_x^m D_t^n a(x,t) \cdot b(x,t) \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n a(x,t)b(x',t')|_{x'=x,t'=t}$$

$$\exp(\delta D_n)a_n \cdot b_n \equiv \exp\left[\delta\left(\frac{\partial}{\partial n} - \frac{\partial}{\partial n'}\right)\right] a(n)b(n')|_{n'=n} = a(n+\delta)b(n-\delta).$$

A BT for (2) is given by [26]

$$[D_t + 2\lambda \sinh(\tfrac{1}{2}D_n)]f_n \cdot f_n' = 0 \qquad \cosh(\tfrac{1}{2}D_n)f_n \cdot f_n' = \lambda f_n f_n'. \tag{3}$$

Here and after, we always denote $f_n(t) = f(n,t) = f(n) = f$ without confusion. Now let $f_0$ be a solution of differential-difference analogue of the KdV equation (2). Suppose that $f_i$ ($i = 1, 2$) is a solution of (2) which is related by $f_0$ under BT (3) with $\lambda_i$, i.e. $f_0 \xrightarrow{\lambda_i} f_i$ ($i = 1, 2$), $f_i \neq 0$ ($i = 0, 1, 2$) and that $f_{12}$ is defined by

$$\cosh(\tfrac{1}{4}D_n)f_0 \cdot f_{12} = k\sinh(\tfrac{1}{4}D_n)f_1 \cdot f_2 \tag{4}$$

(where $k = k(t)$ is some non-zero function of $t$).

From these assumptions, we deduce that

$$\begin{aligned}
0 &= [(\cosh(\tfrac{1}{2}D_n) - \lambda_1)f_0 \cdot f_1]f_2 - [(\cosh(\tfrac{1}{2}D_n) - \lambda_2)f_0 \cdot f_2]f_1 \\
&\overset{(A.1)}{=} f_0(n+\tfrac{1}{2})\exp\left(-\frac{1}{4}\frac{\partial}{\partial n}\right)[\sinh(-\tfrac{1}{4}D_n)f_1 \cdot f_2] \\
&\quad + f_0(n-\tfrac{1}{2})\exp\left(\frac{1}{4}\frac{\partial}{\partial n}\right)[\sinh(\tfrac{1}{4}D_n)f_1 \cdot f_2] + (\lambda_2 - \lambda_1)f_0 f_1 f_2 \\
&\overset{(4)}{=} f_0(n+\tfrac{1}{2})\left[-\frac{1}{k}\exp\left(-\frac{1}{4}\frac{\partial}{\partial n}\right)\cosh(\tfrac{1}{4}D_n)f_0 \cdot f_{12}\right] \\
&\quad + f_0(n-\tfrac{1}{2})\left[\frac{1}{k}\exp\left(\frac{1}{4}\frac{\partial}{\partial n}\right)\cosh(\tfrac{1}{4}D_n)f_0 \cdot f_{12}\right] + (\lambda_2 - \lambda_1)f_0 f_1 f_2
\end{aligned}$$

$$= -\frac{1}{2k} f_0(n+\tfrac{1}{2}) \exp\left(-\frac{1}{4}\frac{\partial}{\partial n}\right) [f_0(n+\tfrac{1}{4}) f_{12}(n-\tfrac{1}{4}) + f_0(n-\tfrac{1}{4}) f_{12}(n+\tfrac{1}{4})]$$

$$+ \frac{1}{2k} f_0(n-\tfrac{1}{2}) \exp\left(\frac{1}{4}\frac{\partial}{\partial n}\right) [f_0(n+\tfrac{1}{4}) f_{12}(n-\tfrac{1}{4}) + f_0(n-\tfrac{1}{4}) f_{12}(n+\tfrac{1}{4})]$$

$$+ (\lambda_2 - \lambda_1) f_0(n) f_1(n) f_2(n)$$

$$= -\frac{1}{2k} f_0(n+\tfrac{1}{2})[f_0(n) f_{12}(n-\tfrac{1}{2}) + f_0(n-\tfrac{1}{2}) f_{12}(n)]$$

$$+ \frac{1}{2k} f_0(n-\tfrac{1}{2})[f_0(n+\tfrac{1}{2}) f_{12}(n) + f_0(n) f_{12}(n+\tfrac{1}{2})]$$

$$+ (\lambda_2 - \lambda_1) f_0(n) f_1(n) f_2(n)$$

$$= f_0(n) \left\{ \frac{1}{2k}[f_0(n-\tfrac{1}{2}) f_{12}(n+\tfrac{1}{2}) - f_0(n+\tfrac{1}{2}) f_{12}(n-\tfrac{1}{2})] \right.$$

$$\left. + (\lambda_2 - \lambda_1) f_1(n) f_2(n) \right\}$$

which implies

$$\sinh(\tfrac{1}{2}D_n) f_0 \cdot f_{12} = k(\lambda_2 - \lambda_1) f_1 f_2. \tag{5}$$

Next we have

$$0 = [(\cosh(\tfrac{1}{2}D_n) - \lambda_1) f_0 \cdot f_1] f_2 + [(\cosh(\tfrac{1}{2}D_n) - \lambda_2) f_0 \cdot f_2] f_1$$

$$\overset{(A.2)}{=} f_0(n+\tfrac{1}{2}) \exp\left(-\frac{1}{4}\frac{\partial}{\partial n}\right) \cosh(\tfrac{1}{4}D_n) f_1 \cdot f_2$$

$$+ f_0(n-\tfrac{1}{2}) \exp\left(\frac{1}{4}\frac{\partial}{\partial n}\right) \cosh(\tfrac{1}{4}D_n) f_1 \cdot f_2 - (\lambda_2 + \lambda_1) f_0 f_1 f_2$$

$$= f_0(n+\tfrac{1}{2}) \exp\left(-\frac{1}{4}\frac{\partial}{\partial n}\right) \left[\cosh(\tfrac{1}{4}D_n) f_1 \cdot f_2 - \frac{\lambda_1 + \lambda_2}{k(\lambda_2 - \lambda_1)} \sinh(\tfrac{1}{4}D_n) f_0 \cdot f_{12}\right]$$

$$+ f_0(n-\tfrac{1}{2}) \exp\left(\frac{1}{4}\frac{\partial}{\partial n}\right) \left[\cosh(\tfrac{1}{4}D_n) f_1 \cdot f_2 - \frac{\lambda_1 + \lambda_2}{k(\lambda_2 - \lambda_1)} \sinh(\tfrac{1}{4}D_n) f_0 \cdot f_{12}\right]$$

$$+ f_0(n) \frac{\lambda_1 + \lambda_2}{k(\lambda_2 - \lambda_1)} \sinh(\tfrac{1}{2}D_n) f_0 \cdot f_{12} - (\lambda_1 + \lambda_2) f_0(n) f_1(n) f_2(n)$$

$$\overset{(5)}{=} f_0(n+\tfrac{1}{2}) \exp\left(-\frac{1}{4}\frac{\partial}{\partial n}\right) \left[\cosh(\tfrac{1}{4}D_n) f_1 \cdot f_2 - \frac{\lambda_1 + \lambda_2}{k(\lambda_2 - \lambda_1)} \sinh(\tfrac{1}{4}D_n) f_0 \cdot f_{12}\right]$$

$$+ f_0(n-\tfrac{1}{2}) \exp\left(\frac{1}{4}\frac{\partial}{\partial n}\right) \left[\cosh(\tfrac{1}{4}D_n) f_1 \cdot f_2 - \frac{\lambda_1 + \lambda_2}{k(\lambda_2 - \lambda_1)} \sinh(\tfrac{1}{4}D_n) f_0 \cdot f_{12}\right]$$

$$= \frac{2}{f_0(n)} \cosh(\tfrac{1}{4}D_n) \left[\cosh(\tfrac{1}{4}D_n) f_1 \cdot f_2 - \frac{\lambda_1 + \lambda_2}{k(\lambda_2 - \lambda_1)} \sinh(\tfrac{1}{4}D_n) f_0 \cdot f_{12}\right]$$

$$\cdot f_0(n+\tfrac{1}{4}) f_0(n-\tfrac{1}{4}).$$

Further we assume that $f_0(n+\epsilon)\xrightarrow{\lambda_i} f_i(n+\epsilon)$, ($i = 1, 2 and \epsilon$ is an arbitrary constant). Similar to the above deduction, we have

$$\cosh(\tfrac{1}{4}D_n)\Big[\cosh(\tfrac{1}{4}D_n)f_1(n+\epsilon)\cdot f_2(n+\epsilon) - \frac{\lambda_1+\lambda_2}{k(\lambda_2-\lambda_1)}\sinh(\tfrac{1}{4}D_n)f_0(n+\epsilon)\cdot f_{12}(n+\epsilon)\Big]\cdot f_0(n+\tfrac{1}{4}+\epsilon)f_0(n-\tfrac{1}{4}+\epsilon) = 0$$

which implies that, by noting $f_0 \neq 0$,

$$\cosh(\tfrac{1}{4}D_n)f_1\cdot f_2 - \frac{\lambda_1+\lambda_2}{k(\lambda_2-\lambda_1)}\sinh(\tfrac{1}{4}D_n)f_0\cdot f_{12} = 0. \tag{6}$$

Thus we have

$$\begin{aligned} -Q_1 f_0 f_2 &\equiv [(\cosh(\tfrac{1}{2}D_n)-\lambda_2)f_0\cdot f_2]f_1 f_{12} - [(\cosh(\tfrac{1}{2}D_n)-\lambda_2)f_1\cdot f_{12}]f_0 f_2 \\ &\overset{(A.3)}{=} 2\{\sinh(\tfrac{1}{4}D_n)[\sinh(\tfrac{1}{4}D_n)f_0\cdot f_{12}]\cdot[\cosh(\tfrac{1}{4}D_n)f_1\cdot f_2] \\ &\quad + \sinh(\tfrac{1}{4}D_n)[\cosh(\tfrac{1}{4}D_n)f_0\cdot f_{12}]\cdot[\sinh(\tfrac{1}{4}D_n)f_1\cdot f_2]\} \\ &\overset{(4)(6)(A.4)}{=} 0 \end{aligned}$$

which implies that

$$[\cosh(\tfrac{1}{2}D_n)-\lambda_2]f_1\cdot f_{12} = 0. \tag{7}$$

Similarly we can prove

$$[\cosh(\tfrac{1}{2}D_n)-\lambda_1]f_2\cdot f_{12} = 0 \tag{8}$$

From $f_{12}[\cosh(\tfrac{1}{2}D_n)-\lambda_1]f_1\cdot f_0 - f_0[\cosh(\tfrac{1}{2}D_n)-\lambda_2]f_1\cdot f_{12} = 0$, similar to the deduction of (5), we can obtain

$$\sinh(\tfrac{1}{2}D_n)f_1\cdot f_2 = \frac{1}{k}(\lambda_1+\lambda_2)f_0 f_{12}. \tag{9}$$

Moreover, we have

$$\begin{aligned} 0 &= \{[D_t + 2\lambda_1\sinh(\tfrac{1}{2}D_n)]f_0\cdot f_1\}f_2 - \{[D_t+2\lambda_2\sinh(\tfrac{1}{2}D_n)]f_0\cdot f_2\}f_1 \\ &\overset{(A.5)}{=} -f_0 D_t f_1\cdot f_2 + (\lambda_1+\lambda_2)\{[\sinh(\tfrac{1}{2}D_n)f_0\cdot f_1]f_2 - [\sinh(\tfrac{1}{2}D_n2)f_0\cdot f_2]f_1\} \\ &\quad + (\lambda_1-\lambda_2)\{[\sinh(\tfrac{1}{2}D_n)f_0\cdot f_1]f_2 + [\sinh(\tfrac{1}{2}D_n)f_0\cdot f_2]f_1\} \\ &\overset{(A.6)(A.7)}{=} -f_0 D_t f_1\cdot f_2 + (\lambda_1+\lambda_2)\Big[f_0(n+\tfrac{1}{2})\exp\Big(-\frac{1}{4}\frac{\partial}{\partial n}\Big)\sinh(-\tfrac{1}{4}D_n)f_1\cdot f_2 \\ &\quad - f_0(n-\tfrac{1}{2})\exp\Big(\frac{1}{4}\frac{\partial}{\partial n}\Big)\sinh(\tfrac{1}{4}D_n)f_1\cdot f_2\Big] \end{aligned}$$

$$+ (\lambda_1 - \lambda_2)\Big[ f_0(n+\tfrac{1}{2}) \exp\left(-\frac{1}{4}\frac{\partial}{\partial n}\right) \cosh(\tfrac{1}{4}D_n) f_1 \cdot f_2$$

$$- f_0(n-\tfrac{1}{2}) \exp\left(\frac{1}{4}\frac{\partial}{\partial n}\right) \cosh(\tfrac{1}{4}D_n) f_1 \cdot f_2\Big]$$

$$\overset{(4)(6)}{=} - f_0 D_t f_1 \cdot f_2 + (\lambda_1 + \lambda_2)\Big[ -\frac{1}{k} f_0(n+\tfrac{1}{2}) \exp\left(-\frac{1}{4}\frac{\partial}{\partial n}\right) \cosh(\tfrac{1}{4}D_n) f_0 \cdot f_{12}$$

$$- \frac{1}{k} f_0(n-\tfrac{1}{2}) \exp\left(\frac{1}{4}\frac{\partial}{\partial n}\right) \cosh(\tfrac{1}{4}D_n) f_0 \cdot f_{12}\Big]$$

$$+ (\lambda_1 - \lambda_2)\Big[ \frac{\lambda_1+\lambda_2}{k(\lambda_2-\lambda_1)} f_0(n+\tfrac{1}{2}) \exp\left(-\frac{1}{4}\frac{\partial}{\partial n}\right) \sinh(\tfrac{1}{4}D_n) f_0 \cdot f_{12}$$

$$- \frac{\lambda_1+\lambda_2}{k(\lambda_2-\lambda_1)} f_0(n-\tfrac{1}{2}) \exp\left(\frac{1}{4}\frac{\partial}{\partial n}\right) \sinh(\tfrac{1}{4}D_n) f_0 \cdot f_{12}\Big]$$

$$= - f_0(n)\Big\{ D_t f_1(n) \cdot f_2(n) + \frac{1}{k}(\lambda_1+\lambda_2)[f_0(n+\tfrac{1}{2}) f_{12}(n-\tfrac{1}{2})$$

$$+ f_0(n-\tfrac{1}{2}) f_{12}(n+\tfrac{1}{2})]\Big\}$$

which implies that

$$D_t f_1 \cdot f_2 + \frac{2(\lambda_1+\lambda_2)}{k} \cosh(\tfrac{1}{2}D_n) f_0 \cdot f_{12} = 0. \tag{10}$$

Since $f_1$ and $f_2$ satisfy equation (2), we have

$$0 = [\cosh(\tfrac{1}{4}D_n) f_2 \cdot f_2] \sinh(\tfrac{1}{4}D_n)[D_t + 2\sinh(\tfrac{1}{2}D_n)] f_1 \cdot f_1$$

$$- [\cosh(\tfrac{1}{4}D_n) f_1 \cdot f_1] \sinh(\tfrac{1}{4}D_n)[D_t + 2\sinh(\tfrac{1}{2}D_n)] f_2 \cdot f_2$$

$$\overset{(A.8)(A.9)}{=} 2\sinh(\tfrac{1}{4}D_n)(D_t f_1 \cdot f_2) \cdot f_1 f_2 + 4\sinh(\tfrac{1}{4}D_n)[\sinh(\tfrac{1}{2}D_n) f_1 \cdot f_2] \cdot [\cosh(\tfrac{1}{2}D_n) f_1 \cdot f_2]$$

$$\overset{(A.9)(A.10)}{=} \frac{4}{k}(\lambda_1+\lambda_2) \sinh(\tfrac{1}{4}D_n)\{f_0 f_{12} \cdot [\cosh(\tfrac{1}{2}D_n) f_1 \cdot f_2]\}$$

$$+ 2D_t[\sinh(\tfrac{1}{4}D_n) f_1 \cdot f_2] \cdot [\cosh(\tfrac{1}{4}D_n) f_1 \cdot f_2]$$

$$\overset{(4)(6)}{=} -\frac{4}{k}(\lambda_1+\lambda_2) \sinh(\tfrac{1}{4}D_n)[\cosh(\tfrac{1}{2}D_n) f_1 \cdot f_2] \cdot f_0 f_{12}$$

$$+ \frac{2(\lambda_1+\lambda_2)}{k^2(\lambda_2-\lambda_1)} D_t[\cosh(\tfrac{1}{4}D_n) f_0 \cdot f_{12}] \cdot [\sinh(\tfrac{1}{4}D_n) f_0 \cdot f_{12}]$$

$$\overset{(A.10)}{=} -\frac{2}{k}(\lambda_1+\lambda_2) \sinh(\tfrac{1}{4}D_n) \left[ \frac{1}{k(\lambda_2-\lambda_1)} D_t f_0 \cdot f_{12} + 2\cosh(\tfrac{1}{2}D_n) f_1 \cdot f_2 \right] \cdot f_0 f_{12}$$

which implies that

$$\frac{1}{k(\lambda_2-\lambda_1)} D_t f_0 \cdot f_{12} + 2\cosh(\tfrac{1}{2}D_n) f_1 \cdot f_2 = c(t) f_0 f_{12} \tag{11}$$

where $c(t)$ is some function of $t$.

Now we set $\widetilde{f}_{12} = \overline{k}(t) f_{12}$ where $\overline{k}(t)$ satisfies

$$\overline{k}_t(t) = (\lambda_2 - \lambda_1) k(t) c(t) \overline{k}(t) \tag{12}$$

i.e.

$$\overline{k}(t) = \exp\left[\int (\lambda_2 - \lambda_1) k(t) c(t)\,\mathrm{d}t\right]. \tag{13}$$

Thus we have from (4)–(10)

$$\cosh(\tfrac{1}{4}D_n) f_0 \cdot \widetilde{f}_{12} = \overline{k}(t) k(t) \sinh(\tfrac{1}{4}D_n) f_1 \cdot f_2 \tag{4'}$$

$$\sinh(\tfrac{1}{2}D_n) f_0 \cdot \widetilde{f}_{12} = \overline{k}(t) k(t) (\lambda_2 - \lambda_1) f_1 f_2 \tag{5'}$$

$$\cosh(\tfrac{1}{4}D_n) f_1 \cdot f_2 - \frac{\lambda_1 + \lambda_2}{\overline{k}(t) k(t) (\lambda_2 - \lambda_1)} \sinh(\tfrac{1}{4}D_n) f_0 \cdot \widetilde{f}_{12} = 0 \tag{6'}$$

$$[\cosh(\tfrac{1}{2}D_n) - \lambda_2] f_1 \cdot \widetilde{f}_{12} = 0 \tag{7'}$$

$$[\cosh(\tfrac{1}{2}D_n) - \lambda_1] f_2 \cdot \widetilde{f}_{12} = 0 \tag{8'}$$

$$\sinh(\tfrac{1}{2}D_n) f_1 \cdot f_2 = \frac{1}{\overline{k}(t) k(t)} (\lambda_1 + \lambda_2) f_0 \widetilde{f}_{12} \tag{9'}$$

$$D_t f_1 \cdot f_2 + \frac{2(\lambda_1 + \lambda_2)}{\overline{k}(t) k(t)} \cosh(\tfrac{1}{2}D_n) f_0 \cdot \widetilde{f}_{12} = 0. \tag{10'}$$

From (11), we get

$$\begin{aligned} &\frac{1}{\overline{k}(t) k(t) (\lambda_2 - \lambda_1)} D_t f_0 \cdot \widetilde{f}_{12} + 2\cosh(\tfrac{1}{2}D_n) f_1 \cdot f_2 \\ &\quad = \frac{1}{k(t)(\lambda_2 - \lambda_1)} D_t f_0 \cdot f_{12} - \frac{\overline{k}_t}{k\overline{k}(\lambda_2 - \lambda_1)} f_0 f_{12} + 2\cosh(\tfrac{1}{2}D_n) f_1 \cdot f_2 \\ &\quad \overset{(11)}{=} [c(t) - \frac{\overline{k}_t}{k\overline{k}(\lambda_2 - \lambda_1)}] f_0 f_{12} = 0. \end{aligned} \tag{11'}$$

By the use of (11′), similar to the deduction of (10), we can show that

$$\begin{aligned} &-\{[D_t + 2\lambda_2 \sinh(\tfrac{1}{2}D_n)] f_1 \cdot \widetilde{f}_{12}\} f_0 \\ &\quad = \{[D_t + 2\lambda_1 \sinh(\tfrac{1}{2}D_n)] f_1 \cdot f_0\} \widetilde{f}_{12} - \{[D_t + 2\lambda_2 \sinh(\tfrac{1}{2}D_n)] f_1 \cdot \widetilde{f}_{12}\} f_0 \\ &\quad = 0 \end{aligned}$$

i.e.

$$[D_t + 2\lambda_2 \sinh(\tfrac{1}{2}D_n)] f_1 \cdot \widetilde{f}_{12} = 0. \tag{14}$$

Similarly, we can show

$$[D_t + 2\lambda_1 \sinh(\tfrac{1}{2}D_n)] f_2 \cdot \widetilde{f}_{12} = 0. \tag{15}$$

Equations (7′), (8′), (14) and (15) imply that $\widetilde{f}_{12}$ is a new solution of differential-difference analogue of the KdV equation (2).

To sum up, we can seek particular solutions of the differential-difference analogue of the KdV equation via the following steps. First choose a given solution $f_0$ of (2). Second from BT (3), we find $f_1$ and $f_2$ such that $f_0(n+\epsilon)\xrightarrow{\lambda_i} f_i(n+\epsilon)$ ($i=1,2$, $\epsilon$ arbitrary constant) and further get a particular solution $f_{12}$ from (4). Finally we substitute $f_{12}$ into (11) and determine $c(t)$ .Then $\widetilde{f}_{12}=\overline{k}(t)f_{12}$ is a new solution of (2), where $\overline{k}(t)$ is given by (13).

In what follows, we give two illustrative examples.

(i) Choose $f_0=1$. It is easily verified that

$$\begin{array}{ccccc} & \cosh(\frac{1}{2}p_1) \nearrow & e^{\eta_1}+e^{-\eta_1} & \searrow \cosh(\frac{1}{2}p_2) & \\ 1 & & & & \dfrac{1}{\cosh(\frac{1}{2}p_1)+\cosh(\frac{1}{2}p_2)}[\sinh(\tfrac{1}{2}(p_1-p_2))e^{\eta_1+\eta_2} \\ & \cosh(\frac{1}{2}p_2) \searrow & e^{\eta_2}+e^{-\eta_2} & \nearrow \cosh(\frac{1}{2}p_1) & \end{array}$$

$$+\sinh(\tfrac{1}{2}(p_1+p_2))e^{\eta_1-\eta_2}-\sinh(\tfrac{1}{2}(p_1+p_2))e^{\eta_2-\eta_1}+\sinh(\tfrac{1}{2}(p_2-p_1))e^{\eta_1-\eta_2}].$$

Therefore

$$\frac{1}{\cosh(\frac{1}{2}p_1)+\cosh(\frac{1}{2}p_2)}[\sinh(\tfrac{1}{2}(p_1-p_2))e^{\eta_1+\eta_2}+\sinh(\tfrac{1}{2}(p_1+p_2))e^{\eta_1-\eta_2}$$
$$-\sinh(\tfrac{1}{2}(p_1+p_2))e^{\eta_2-\eta_1}+\sinh(\tfrac{1}{2}(p_2-p_1))e^{-\eta_1-\eta_2}]$$

is a solution of (2), where $\eta_i=p_in-\sinh(p_i)t+\eta_i^0$ and $p_i,\eta_i^0$ are constants ($i=1,2$).

(ii) It is easily verified that

$$\begin{array}{ccccc} & 1 \nearrow & t-n & \searrow \cosh(\frac{1}{2}p) & \\ 1 & & & & -\dfrac{1}{1+\cosh(\frac{1}{2}p)}[2(t-n)\sinh(\tfrac{1}{2}p)\sinh(\eta) \\ & \cosh(\frac{1}{2}p) \searrow & e^{\eta}+e^{-\eta} & \nearrow 1 & \end{array}$$

$$+\cosh(\tfrac{1}{2}p)\cosh(\eta)].$$

Therefore $-1/(1+\cosh(\frac{1}{2}p))[2(t-n)\sinh(\frac{1}{2}p)\sinh(\eta)+\cosh(\frac{1}{2}p)\cosh(\eta)]$ is also a solution of (2), where $\eta=pn-\sinh(p)t+\eta^0$ and $p,\eta^0$ are constants.

## 3. Nonlinear superposition formula of two-dimensional Toda equation

In 1915, Darboux introduced the nonlinear differential-difference equation [28]

$$h_{n+1}+h_{n-1}=2h_n+\frac{\partial^2}{\partial x\partial y}(\ln h_n). \tag{16}$$

By introducing a new variable $Q_n$ related to $h_n$ by

$$h_n=\exp(Q_{n-1}-Q_n).$$

We can represent (16) in the form

$$\frac{\partial^2}{\partial x\partial y}Q_n=\exp(Q_{n-1}-Q_n)-\exp(Q_n-Q_{n+1}). \tag{17}$$

We refer to (17) as the two-dimensional Toda lattice equation. So far, much research on (17) has been conducted. For example, Mikhailov [29] established the integrability of (17). In addition a Darboux transformation for (17) has been introduced [30]. In this section, we shall establish a nonlinear superposition formula for the two-dimensional Toda lattice equation. To this end, we introduce $h_n = (\partial^2/\partial x \partial y) \ln f_n$ (or $Q_n = \ln f_n/f_{n+1}$), then (16) (or (17)) can be reduced to

$$[D_x D_y - 4\sinh^2(\tfrac{1}{2}D_n)] f_n \cdot f_n = 0. \tag{18}$$

Concerning (18), we have the following results.

*Proposition 1.* A BT for (18) is

$$(D_x + \lambda^{-1}\exp(-D_n) + \mu) f_n \cdot f_n' = 0 \tag{19a}$$

$$[D_y \exp(-\tfrac{1}{2}D_n) - \lambda \exp(\tfrac{1}{2}D_n) + \gamma \exp(-\tfrac{1}{2}D_n)] f_n \cdot f_n' = 0 \tag{19b}$$

where $\lambda, \mu, \gamma$ are arbitrary constants.

*Proof.* Let $f_n$ and $f_n'$ be two solutions of (18). If we can find two equations which relate $f_n$ and $f_n'$, and satisfy

$$P \equiv f_n'^2[D_x D_y - 4\sinh^2(\tfrac{1}{2}D_n)] f_n \cdot f_n - f_n^2[D_x D_y - 4\sinh^2(\tfrac{1}{2}D_n)] f_n' \cdot f_n' = 0.$$

This is then a BT. Here we show that (19*a*) and (19*b*) indeed provide a BT for (18).

Making use of (A.11)–(A.13), (19*a*) and (19*b*), $P$ can be rewritten as

$$\begin{aligned}
P &= 2D_y(D_x f_n \cdot f_n') \cdot f_n f_n' - 4\sinh(\tfrac{1}{2}D_n)[\exp(\tfrac{1}{2}D_n) f_n \cdot f_n'] \cdot [\exp(-\tfrac{1}{2}D_n) f_n \cdot f_n'] \\
&= -2\lambda^{-1} D_y[\exp(-D_n) f_n \cdot f_n'] \cdot f_n f_n' \\
&\quad - 4\sinh(\tfrac{1}{2}D_n)[\exp(\tfrac{1}{2}D_n) f_n \cdot f_n'] \cdot [\exp(-\tfrac{1}{2}D_n) f_n \cdot f_n'] \\
&= 4\sinh(\tfrac{1}{2}D_n)\{[\lambda^{-1} D_y \exp(-\tfrac{1}{2}D_n) - \exp(\tfrac{1}{2}D_n)] f_n \cdot f_n'\} \cdot [\exp(-\tfrac{1}{2}D_n) f_n \cdot f_n'] \\
&= 0.
\end{aligned}$$

Thus we have completed the proof of the proposition 1. □

In the following, we always denote $f_n(x, y) = f(n, x, y) = f(n) = f$ without confusion.

*Proposition 2.* Let $f_0$ be a solution of (18) and suppose that $f_i$ $(i = 1, 2)$ is a solution of (18), which is related by $f_0$ under BT (19) with $(\lambda_i, \mu_i, \gamma_i)$, i.e. $f_0 \xrightarrow{(\lambda_i, \mu_i, \gamma_i)} f_i$ $(i = 1, 2)$, $\lambda_1\lambda_2 \neq 0$, $f_j \neq 0$ $(j = 0, 1, 2)$. Then $f_{12}$ defined by

$$\exp(-\tfrac{1}{2}D_n) f_0 \cdot f_{12} = k[\lambda_1 \exp(-\tfrac{1}{2}D_n) - \lambda_2 \exp(\tfrac{1}{2}D_n)] f_1 \cdot f_2 \qquad (k \text{ is a non-zero constant}) \tag{20}$$

is a new solution which is related by $f_1$ and $f_2$ under BT (19) with parameters $(\lambda_2, \mu_2, \gamma_2)$, $(\lambda_1, \mu_1, \gamma_1)$ respectively.

*Proof.* It suffices to show that

$$(D_x + \lambda_2^{-1}\exp(-D_n) + \mu_2) f_1 \cdot f_{12} = 0$$
$$(D_x + \lambda_1^{-1}\exp(-D_n) + \mu_1) f_2 \cdot f_{12} = 0$$
$$[D_y \exp(-\tfrac{1}{2}D_n) - \lambda_2 \exp(\tfrac{1}{2}D_n) + \gamma_2 \exp(-\tfrac{1}{2}D_n)] f_1 \cdot f_{12} = 0$$
$$[D_y \exp(-\tfrac{1}{2}D_n) - \lambda_1 \exp(\tfrac{1}{2}D_n) + \gamma_1 \exp(-\tfrac{1}{2}D_n)] f_2 \cdot f_{12} = 0.$$

By use of (A.5) and (A.20), we have

$$\begin{aligned}
0 &= [(D_x + \lambda_1^{-1}\exp(-D_n) + \mu_1) f_0 \cdot f_1] f_2 - [(D_x + \lambda_2^{-1}\exp(-D_n) + \mu_2) f_0 \cdot f_2] f_1 \\
&= f_0(n)[-D_x f_1 \cdot f_2 + (\mu_1 - \mu_2) f_1 f_2] \\
&\quad + \frac{1}{\lambda_1\lambda_2} \exp\left(\frac{1}{2}\frac{\partial}{\partial n}\right) \{ f_0(n - \tfrac{3}{2})[\lambda_2 \exp(\tfrac{1}{2}D_n) - \lambda_1 \exp(-\tfrac{1}{2}D_n)] f_1(n) \cdot f_2(n) \} \\
&= f_0(n)[-D_x f_1 \cdot f_2 + (\mu_1 - \mu_2) f_1 f_2] \\
&\quad - \frac{1}{k\lambda_1\lambda_2} \exp\left(\frac{1}{2}\frac{\partial}{\partial n}\right) [ f_0(n - \tfrac{3}{2}) \exp(-\tfrac{1}{2}D_n) f_0 \cdot f_{12}] \\
&= f_0(n)\left[ -D_x f_1 \cdot f_2 + (\mu_1 - \mu_2) f_1 f_2 - \frac{1}{k\lambda_1\lambda_2} \exp(-D_n) f_0 \cdot f_{12} \right]
\end{aligned}$$

which implies that

$$- D_x f_1(n) \cdot f_2(n) + (\mu_1 - \mu_2) f_1(n) f_2(n) - \frac{1}{k\lambda_1\lambda_2} \exp(-D_n) f_0(n) \cdot f_{12}(n) = 0. \qquad (21)$$

And,

$$\begin{aligned}
0 &= \left\{ \exp\left(-\frac{\partial}{\partial n}\right) [\lambda_1 D_x + \exp(-D_n) + \lambda_1\mu_1] f_0 \cdot f_1 \right\} f_2(n) \\
&\quad - \left\{ \exp\left(-\frac{\partial}{\partial n}\right) [\lambda_2 D_x + \exp(-D_n) + \lambda_2\mu_2] f_0 \cdot f_2 \right\} f_1(n) \\
&= \lambda_1 f_{0x}(n-1) f_1(n-1) f_2(n) - \lambda_1 f_0(n-1) f_{1x}(n-1) f_2(n) \\
&\quad - \lambda_2 f_{0x}(n-1) f_2(n-1) f_1(n) + \lambda_2 f_0(n-1) f_{2x}(n-1) f_1(n) \\
&\quad + f_0(n-1)[\lambda_1\mu_1 f_1(n-1) f_2(n) - \lambda_2\mu_2 f_1(n) f_2(n-1)] \\
&= f_{0x}(n-1) \exp\left(-\frac{1}{2}\frac{\partial}{\partial n}\right) [\lambda_1 \exp(-\tfrac{1}{2}D_n) - \lambda_2 \exp(\tfrac{1}{2}D_n)] f_1(n) \cdot f_2(n) \\
&\quad + f_0(n-1)[-\lambda_1 f_{1x}(n-1) f_2(n) + \lambda_2 f_1(n) f_{2x}(n-1) \\
&\quad + \lambda_1\mu_1 f_1(n-1) f_2(n) - \lambda_2\mu_2 f_1(n) f_2(n-1)] \\
&\overset{(20)}{=} f_{0x}(n-1) \exp\left(-\frac{1}{2}\frac{\partial}{\partial n}\right) \left[ \frac{1}{k} \exp(-\tfrac{1}{2}D_n) f_0 \cdot f_{12} \right] \\
&\quad + f_0(n-1)[-\lambda_1 f_{1x}(n-1) f_2(n) + \lambda_2 f_1(n) f_{2x}(n-1) \\
&\quad + \lambda_1\mu_1 f_1(n-1) f_2(n) - \lambda_2\mu_2 f_1(n) f_2(n-1)]
\end{aligned}$$

$$= f_0(n-1)\left\{\frac{1}{2k}D_x f_0(n-1)\cdot f_{12}(n) + \frac{1}{2k}[\exp(-\tfrac{1}{2}\frac{\partial}{\partial n})\exp(-\tfrac{1}{2}D_n)f_0\cdot f_{12}]_x\right.$$
$$- \lambda_1 f_{1x}(n-1)f_2(n) + \lambda_2 f_1(n)f_{2x}(n-1)$$
$$\left. + \lambda_1\mu_1 f_1(n-1)f_2(n) - \lambda_2\mu_2 f_1(n)f_2(n-1)\right\}$$
$$\stackrel{(20)}{=} f_0(n-1)\left[\frac{1}{2k}D_x f_0(n-1)\cdot f_{12}(n) - \tfrac{1}{2}\lambda_1 D_x f_1(n-1)\cdot f_2(n)\right.$$
$$\left. - \tfrac{1}{2}\lambda_2 D_x f_1(n)\cdot f_2(n-1) + \lambda_1\mu_1 f_1(n-1)f_2(n) - \lambda_2\mu_2 f_1(n)f_2(n-1)\right]$$

which implies that

$$\frac{1}{2k}D_x\exp(-\tfrac{1}{2}D_n)f_0(n)\cdot f_{12}(n) - \tfrac{1}{2}\lambda_1 D_x\exp(-\tfrac{1}{2}D_n)f_1(n)\cdot f_2(n)$$
$$- \tfrac{1}{2}\lambda_2 D_x\exp(\tfrac{1}{2}D_n)f_1(n)\cdot f_2(n)$$
$$+ \lambda_1\mu_1\exp(-\tfrac{1}{2}D_n)f_1(n)\cdot f_2(n) - \lambda_2\mu_2\exp(\tfrac{1}{2}D_n)f_1(n)\cdot f_2(n)$$
$$= 0. \tag{22}$$

Thus we have

$$\{[\lambda_2 D_x + \exp(-D_n) + \lambda_2\mu_2]f_1\cdot f_{12}\}f_0(n-1)$$
$$= \lambda_2\mu_2 f_1(n)f_{12}(n)f_0(n-1) + \lambda_2 f_{1x}(n)\exp\left(-\frac{1}{2}\frac{\partial}{\partial n}\right)\exp(-\tfrac{1}{2}D_n)f_0\cdot f_{12}$$
$$- \lambda_2 f_1(n)f_{12x}(n)f_0(n-1) + f_1(n-1)\exp(-D_n)f_0\cdot f_{12}$$
$$\stackrel{(20),(21)}{=} f_1(n)[\lambda_2\mu_2 f_{12}(n)f_0(n-1) - \lambda_2 f_{12x}(n)f_0(n-1)]$$
$$+ k\lambda_2 f_{1x}(n)\exp\left(-\frac{1}{2}\frac{\partial}{\partial n}\right)[\lambda_1\exp(-\tfrac{1}{2}D_n) - \lambda_2\exp(\tfrac{1}{2}D_n)]f_1\cdot f_2$$
$$+ f_(n-1)k\lambda_1\lambda_2[-D_x f_1\cdot f_2 + (\mu_1-\mu_2)f_1 f_2]$$
$$= f_1(n)[\lambda_2\mu_2 f_{12}(n)f_0(n-1) - \lambda_2 f_{12x}(n)f_0(n-1)$$
$$-k\lambda_2^2 f_{1x}(n)f_2(n-1) + k\lambda_1\lambda_2 f_1(n-1)f_{2x}(n) + k\lambda_1\lambda_2(\mu_1-\mu_2)f_1(n-1)f_2(n)]$$
$$= \lambda_2 f_1(n)\left[\mu_2\exp\left(-\frac{1}{2}\frac{\partial}{\partial n}\right)\exp(-\tfrac{1}{2}D_n)f_0\cdot f_{12}\right.$$
$$+ \tfrac{1}{2}D_x f_0(n-1)\cdot f_{12}(n) - \tfrac{1}{2}\left(\exp\left(-\frac{1}{2}\frac{\partial}{\partial n}\right)\exp(-\tfrac{1}{2}D_n)f_0\cdot f_{12}\right)_x$$
$$\left. - k\lambda_2 f_{1x}(n)f_2(n-1) + k\lambda_1 f_1(n-1)f_{2x}(n) + k\lambda_1(\mu_1-\mu_2)f_1(n-1)f_2(n)\right]$$
$$\stackrel{(20)}{=} \lambda_2 f_1(n)[\tfrac{1}{2}D_x f_0(n-1)\cdot f_{12}(n) - \tfrac{1}{2}k\lambda_1 D_x f_1(n-1)\cdot f_2(n)$$
$$- \tfrac{1}{2}k\lambda_2 D_x f_1(n)\cdot f_2(n-1) + k\lambda_1\mu_1 f_1(n-1)f_2(n) - \lambda_2\mu_2 f_1(n)f_2(n-1)]$$
$$\stackrel{(22)}{=} 0$$

which implies that

$$(D_x + \lambda_2^{-1}\exp(-D_n) + \mu_2) f_1 \cdot f_{12} = 0.$$

Similarly we have

$$(D_x + \lambda_1^{-1}\exp(-D_n) + \mu_1) f_2 \cdot f_{12} = 0.$$

Finally, since $f_1$ and $f_2$ are two solutions of (18), we have

$$\begin{aligned}
0 &= f_2^2[D_xD_y - 4\sinh^2(\tfrac{1}{2}D_n)]f_1\cdot f_1 - f_1^2[D_xD_y - 4\sinh^2(\tfrac{1}{2}D_n)]f_2\cdot f_2\\
&\overset{(A.11),(A.12)}{=} 2D_y(D_xf_1\cdot f_2)\cdot f_1f_2 - 4\sinh(\tfrac{1}{2}D_n)[\exp(\tfrac{1}{2}D_n)f_1\cdot f_2]\cdot[\exp(-\tfrac{1}{2}D_n)f_1\cdot f_2]\\
&\overset{(21),(A.5)}{=} -\frac{2}{k\lambda_1\lambda_2}D_y[\exp(-D_n)f_0\cdot f_{12}]\cdot f_1f_2\\
&\quad + \frac{4}{\lambda_2}\sinh(\tfrac{1}{2}D_n)\{[\lambda_1\exp(-\tfrac{1}{2}D_n) - \lambda_2\exp(\tfrac{1}{2}D_n)]f_1\cdot f_2\}\cdot[\exp(-\tfrac{1}{2}D_n)f_1\cdot f_2]\\
&\overset{(20)}{=} -\frac{2}{k\lambda_1\lambda_2}\{[D_yf_0(n-1)\cdot f_1(n)]f_2(n)f_{12}(n+1)\\
&\quad - f_0(n-1)f_1(n)D_yf_2(n)\cdot f_{12}(n+1)\}\\
&\quad + \frac{4}{k\lambda_2}\sinh(\tfrac{1}{2}D_n)[\exp(-\tfrac{1}{2}D_n)f_0\cdot f_{12}]\cdot[\exp(-\tfrac{1}{2}D_n)f_1\cdot f_2]\\
&= -\frac{2}{k\lambda_1\lambda_2}f_2(n)f_{12}(n+1)\exp\left(-\frac{1}{2}\frac{\partial}{\partial n}\right)D_y\exp(-\tfrac{1}{2}D_n)f_0(n)\cdot f_1(n)\\
&\quad + \frac{2}{k\lambda_1\lambda_2}f_0(n-1)f_1(n)\exp\left(\frac{1}{2}\frac{\partial}{\partial n}\right)D_y\exp(-\tfrac{1}{2}D_n)f_2(n)\cdot f_{12}(n)\\
&\quad + \frac{2}{k\lambda_2}[f_0(n)f_{12}(n+1)f_1(n-1)f_2(n) - f_0(n-1)f_{12}(n)f_1(n)f_2(n+1)]\\
&= \frac{2}{k\lambda_1\lambda_2}f_0(n-1)f_1(n)\Bigg[\exp\left(\frac{1}{2}\frac{\partial}{\partial n}\right)D_y\exp(-\tfrac{1}{2}D_n)f_2(n)\cdot f_{12}(n)\\
&\quad + \gamma_1f_2(n)f_{12}(n+1) - \lambda_1f_2(n+1)f_{12}(n)\Bigg]
\end{aligned}$$

which implies that

$$[D_y\exp(-\tfrac{1}{2}D_n) - \lambda_1\exp(\tfrac{1}{2}D_n) + \gamma_1\exp(-\tfrac{1}{2}D_n)]f_2\cdot f_{12} = 0.$$

Similarly, we can show that

$$[D_y\exp(-\tfrac{1}{2}D_n) - \lambda_2\exp(\tfrac{1}{2}D_n) + \gamma_2\exp(-\tfrac{1}{2}D_n)]f_1\cdot f_{12} = 0.$$

Thus we have completed the proof of proposition 2. □

We can also obtain some particular solutions from BT (19) and nonlinear superposition formula (20).

## 4. Conclusion and discussion

In this paper, we have given nonlinear superposition formulae for the differential-difference analogue of the KdV equation and the two-dimensional Toda equation. As an illustrative application of the obtained result, some particular solutions of the differential-difference analogue of the KdV equation have been obtained.

It is noted that when $x = y = t$ in (18), (18) reduces to the one-dimensional Toda equation.

$$[D_t^2 - 4\sinh^2(\tfrac{1}{2}D_n)]f_n \cdot f_n = 0. \tag{23}$$

Thus we immediately obtain the coressponding BT from (19), which generalizes the results in [27, 31], and further a nonlinear superposition formula of one-dimensional Toda equation (23). It is also noted that a BT and nonlinear superposition formula of one-dimensional Toda equation were given in [20, 21]. It is easily proved that (19) and (20) with $x = y = t$ can be reduced to those in [20, 21]. Furthermore, we note that Hirota and Satsuma [3] presented another BT for (23)

$$\begin{aligned} &[D_t + 2\lambda \sinh(\tfrac{1}{2}D_n)]f_n' \cdot f_n' = 0 \\ &\cosh(\tfrac{1}{2}D_n)f_n \cdot f_n' = \lambda f_n f_n' \end{aligned} \tag{24}$$

which is of the same form (3), and derived the corresponding nonlinear superposition formula

$$\cosh(\tfrac{1}{4}D_n)f_0 \cdot f_{12} = k\sinh(\tfrac{1}{4}D_n)f_1 \cdot f_2 \tag{25}$$

from the commutability of BT. Now, similar to the deduction of section 2, we can prove nonlinear superposition formula (25) rigorously. The unique difference lies in the deduction of (11). This can be overcome by noticing that

$$\begin{aligned} 0 &= f_2^2[D_t^2 - 4\sinh^2(\tfrac{1}{2}D_n)]f_1 \cdot f_1 - f_1^2[D_t^2 - 4\sinh^2(\tfrac{1}{2}D_n)]f_2 \cdot f_2 \\ &\overset{(A.14),(A.15)}{=} 2D_t(D_t f_1 \cdot f_2) \cdot f_1 f_2 - 8\sinh(\tfrac{1}{2}D_n)[\sinh(\tfrac{1}{2}D_n)f_1 \cdot f_2] \cdot [\cosh(\tfrac{1}{2}D_n)f_1 \cdot f_2] \\ &= -\frac{4(\lambda_1+\lambda_2)}{k^2(\lambda_2-\lambda_1)}D_t[\cosh(\tfrac{1}{2}D_n)f_0 \cdot f_{12}] \cdot [\sinh(\tfrac{1}{2}D_n)f_0 \cdot f_{12}] \\ &\quad - 8\sinh(\tfrac{1}{2}D_n)\frac{\lambda_1+\lambda_2}{k}f_0 f_{12} \cdot [\cosh(\tfrac{1}{2}D_n)f_1 \cdot f_2] \\ &\overset{(A.10)}{=} \frac{4(\lambda_1+\lambda_2)}{k}\sinh(\tfrac{1}{2}D_n)\left[\frac{1}{k(\lambda_2-\lambda_1)}D_t f_0 \cdot f_{12} + 2\cosh(\tfrac{1}{2}D_n)f_1 \cdot f_2\right] \cdot f_0 f_{12}. \end{aligned}$$

Finally, it would be of interest to note that the one-dimensional Toda equation (23) has two different nonlinear superposition formulae. Just as for the differential-difference analogue of the KdV equation, from (20) and (25) and their corresponding Backlund transformations respectively, we can get both the following solutions of the one-dimensional Toda equation

$$\begin{aligned} f_n &= \sinh(\tfrac{1}{2}(p_1 - p_2))e^{\eta_1+\eta_2} + \sinh(\tfrac{1}{2}(p_1+p_2))e^{\eta_1-\eta_2} - \sinh(\tfrac{1}{2}(p_1+p_2))e^{\eta_2-\eta_1} \\ &\quad + \sinh(\tfrac{1}{2}(p_2-p_1))e^{-\eta_1-\eta_2} \end{aligned}$$

with $\eta_i = p_i n - \sinh(p_i)t + \eta_i^0$, $p_i, \eta_i^0$ are constants ($i = 1, 2$) and

$$f_n = 2(t-n)\sinh(\tfrac{1}{2}p)\sinh(\eta) + \cosh(\tfrac{1}{2}p)\cosh(\eta)$$

with $\eta = pn - \sinh(p)t + \eta^0$ and $p, \eta^0$ are constants. It seems that BT (24) is not a limiting case of BT (19) with $x = y = t$. The differences between (24), (25) and (19), (20) with $x = y = t$ remains to be studied further.

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## Appendix

The following bilinear operator identities hold for arbitrary functions $a$, $b$, $c$ and $d$:

$$[\cosh(\epsilon D_n)a\cdot b]c-[\cosh(\epsilon D_n)a\cdot c]b$$
$$=a(n+\epsilon)\exp\left(-\frac{1}{2}\epsilon\frac{\partial}{\partial n}\right)\sinh(-\tfrac{1}{2}\epsilon D_n)b\cdot c+a(n-\epsilon)\times\exp\left(\frac{1}{2}\epsilon\frac{\partial}{\partial n}\right)\sinh(\tfrac{1}{2}\epsilon D_n)b\cdot c \quad \text{(A.1)}$$

$$[\cosh(\epsilon D_n)a\cdot b]c+[\cosh(\epsilon D_n)a\cdot c]b$$
$$=a(n+\epsilon)\exp\left(-\frac{1}{2}\epsilon\frac{\partial}{\partial n}\right)\cosh(\tfrac{1}{2}\epsilon D_n)b\cdot c+a(n-\epsilon)\times\exp\left(\frac{1}{2}\epsilon\frac{\partial}{\partial n}\right)\cosh(\tfrac{1}{2}\epsilon D_n)b\cdot c \quad \text{(A.2)}$$

$$[\cosh(2\epsilon D_n)a\cdot b]cd-[\cosh(2\epsilon D_n)c\cdot d]ab$$
$$=2\{\sinh(\epsilon D_n)[\sinh(\epsilon D_n)a\cdot d]\cdot[\cosh(\epsilon D_n)c\cdot b]+\sinh(\epsilon D_n)[\cosh(\epsilon D_n)a\cdot d]\cdot[\sinh(\epsilon D_n)c\cdot b]\} \quad \text{(A.3)}$$

$$\sinh(\epsilon D_n)a\cdot a=0 \quad \text{(A.4)}$$

$$(D_ta\cdot b)c-(D_ta\cdot c)b=-aD_tb\cdot c \quad \text{(A.5)}$$

$$[\sinh(\epsilon D_n)a\cdot b]c-[\sinh(\epsilon D_n)a\cdot c]b$$
$$=a(n+\epsilon)\exp\left(-\frac{1}{2}\epsilon\frac{\partial}{\partial n}\right)\sinh(-\tfrac{1}{2}\epsilon D_n)b\cdot c-a(n-\epsilon)\times\exp\left(\frac{1}{2}\epsilon\frac{\partial}{\partial n}\right)\sinh(\tfrac{1}{2}\epsilon D_n)b\cdot c \quad \text{(A.6)}$$

$$[\sinh(\epsilon D_n)a\cdot b]c+[\sinh(\epsilon D_n)a\cdot c]b$$
$$=a(n+\epsilon)\exp\left(-\frac{1}{2}\epsilon\frac{\partial}{\partial n}\right)\cosh(\tfrac{1}{2}\epsilon D_n)b\cdot c-a(n-\epsilon)\times\exp\left(\frac{1}{2}\epsilon\frac{\partial}{\partial n}\right)\cosh(\tfrac{1}{2}\epsilon D_n)b\cdot c \quad \text{(A.7)}$$

$$[\cosh(\tfrac{1}{4}D_n)b\cdot b][D_t\sinh(\tfrac{1}{4}D_n)a\cdot a]-[\cosh(\tfrac{1}{4}D_n)a\cdot a][D_t\sinh(\tfrac{1}{4}D_n)b\cdot b]$$
$$=2\sinh(\tfrac{1}{4}D_n)(D_ta\cdot b)\cdot ab \quad \text{(A.8)}$$

$$[\cosh(\tfrac{1}{4}D_n)b\cdot b][\sinh(\tfrac{1}{4}D_n)\sinh(\tfrac{1}{2}D_n)a\cdot a]-[\cosh(\tfrac{1}{4}D_n)a\cdot a][\sinh(\tfrac{1}{4}D_n)\sinh(\tfrac{1}{2}D_nb\cdot b]$$
$$=2\sinh(\tfrac{1}{4}D_n)[\sinh(\tfrac{1}{2}D_n)a\cdot b]\cdot[\cosh(\tfrac{1}{2}D_n)a\cdot b] \tag{A.9}$$

$$D_t[\sinh(\epsilon D_n)a\cdot b]\cdot[\cosh(\epsilon D_n)a\cdot b]=\sinh(\epsilon D_n)(D_ta\cdot b)\cdot ab \tag{A.10}$$

$$(D_xD_ya\cdot a)b^2-a^2D_xD_yb\cdot b=2D_y(D_xa\cdot b)\cdot ab \tag{A.11}$$

$$[\sinh^2(\epsilon D_n)a\cdot a]b^2-a^2[\sinh^2(\epsilon D_n)b\cdot b]$$
$$=\sinh(\epsilon D_n)[\exp(\epsilon D_n)a\cdot b]\cdot[\exp(-\epsilon D_n)a\cdot b] \tag{A.12}$$

$$D_y[\exp(2\epsilon D_n)a\cdot b]\cdot ab=\sinh(\epsilon D_n)[D_y\exp(\epsilon D_n)a\cdot b]\cdot[\exp(\epsilon D_n)a\cdot b] \tag{A.13}$$

$$(D_t^2a\cdot a)b^2-a^2D_t^2b\cdot b=2D_t(D_ta\cdot b)\cdot ab \tag{A.14}$$

$$[\sinh^2(\epsilon D_n)a\cdot a]b^2-a^2[\sinh^2(\epsilon D_n)b\cdot b$$
$$=2\sinh(\epsilon D_n)[\sinh(\epsilon D_n)a\cdot b]\cdot[\cosh(\epsilon D_n)a\cdot b]. \tag{A.15}$$

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